Linear Algebra II

11/04/2012, Wednesday, 9:00-12:00

1 (8+7=15 pt) **Inner product spaces**

(a) Consider the vector space $\mathbb{R}^n$. Show that $\langle x, y\rangle = x^T My$ is an inner product if and only if $M \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

(b) Let V be an inner product space and let $\|v\| = \sqrt{\langle v, v\rangle}$. Prove that

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$.

2 (2+2+5+6=15 pt) **Orthogonal matrices**

Let $A \in \mathbb{R}^{n\times n}$.

(a) Show that if $(I + A)$ is nonsingular then $(I - A)(I + A)^{-1} = (I + A)^{-1}(I - A)$.

(b) Show that if $A = -A^T$ then $(I + A)$ is nonsingular.

(c) Show that if $A = -A^T$ then $(I - A)(I + A)^{-1}$ is an orthogonal matrix.

(d) Show that if A is orthogonal and $(I + A)$ is nonsingular then $B = -B^T$ where $B = (I - A)(I + A)^{-1}$

3 (8+7=15 pt) **Diagonalization and positive definite matrices**

Let

$$A = \begin{bmatrix} a & b & 0 \\ c & d & c \\ 0 & b & a \end{bmatrix}$$

where a, b, c, and d are real numbers.

(a) For which values of (a, b, c, d) is the matrix A unitarily diagonalizable?

(b) For which values of (a, b, c, d) is the matrix A positive definite? (Warning: The matrix A is not necessarily symmetric!)

4 (8+7=15 pt) **Cayley-Hamilton theorem**

(a) Let $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Show that the subspace $\operatorname{span}\{v, Av, \ldots, A^{n-1}v\}$ is invariant under A.

(b) Let

$$M = \begin{bmatrix} 0 & 0 & -1+a \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}.$$

Find $(M+I)^{3000}$.

5 (15 pt) **Singular value decomposition**

Find a singular value decomposition for the matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

and its best rank 1 approximation in the Frobenius norm.

6 (15 pt) **Jordan canonical form**

Put the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

into Jordan canonical form.

Hint: Note that $(a \pm 1)^3 = a^3 \pm 3a^2 + 3a \pm 1$.

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10 pt gratis

(1)

(a) A functional $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product if

(i) $\langle x, x \rangle \geq 0 \quad \forall x \in \mathbb{R}^n$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(ii) $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in \mathbb{R}^n$

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{R} \quad x, y, z \in \mathbb{R}^n$.

<u>if</u>: Since M is positive definite, we have

$$\langle x, x \rangle = x^T M x > 0 \quad \forall \, 0 \neq x \in \mathbb{R}^n.$$

Hence, (i) holds. Note that

$$\langle x, y \rangle = x^T M y = (x^T M y)^T = y^T M^T x = y^T M x = \langle y, x \rangle$$

due to symmetry. Therefore, (ii) holds too. Finally, note that

$$\langle \alpha x + \beta y, z \rangle = (\alpha x + \beta y)^T M z = \alpha \, x^T M z + \beta \, y^T M z$$

$$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle.$$

Consequently, (iii) also holds and $\langle \cdot, \cdot \rangle$ is an inner product.

only if: Since $\langle x,y\rangle = x^T M y$ is an inner product, it follows from (i) that

$$\langle x,x\rangle = x^T M x > 0 \qquad \forall\, 0 \neq x \in \mathbb{R}^n$$

Hence, M is a positive definite matrix. From (ii), we have

$$x^T M y = \langle x,y\rangle = \langle y,x\rangle = y^T M x = (y^T M x)^T = x^T M^T y \quad \forall\, x,y \in \mathbb{R}^n$$

Therefore, $0 = x^T M y - x^T M^T y = x^T (M - M^T) y$ for all $x,y \in \mathbb{R}^n$.

Then, we can conclude that $M = M^T$, i.e. M is symmetric.

(b) $$\|x-y\|^2 + \|x+y\|^2 = \langle x-y, x-y\rangle + \langle x+y, x+y\rangle$$

$$= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$

$$+ \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= 2\langle x,x\rangle + 2\langle y,y\rangle$$

$$= 2\|x\|^2 + 2\|y\|^2.$$

②

ⓐ Note that

$$(I+A)(I-A)(I+A)^{-1}(I+A) = (I+A)(I-A) = I - A^2$$

and

$$(I+A)(I+A)^{-1}(I-A)(I+A) = (I-A)(I+A) = I - A^2.$$

Since $(I+A)$ is nonsingular, we get

$$(I-A)(I+A)^{-1} = (I+A)^{-1}(I-A).$$

ⓑ $A = -A^T \overset{?}{\Longrightarrow} (I+A)$ is nonsingular.

Let x be such that

$$0 = (I+A)x.$$

Then, we have

$$0 = x^T(I+A)x = x^Tx + x^TAx \qquad (*)$$

By transposing, we get

$$0 = x^Tx + x^TA^Tx = x^Tx - x^TAx \qquad (\text{since } A = -A^T) \qquad (**)$$

By adding $(**)$ to $(*)$, we obtain

$$0 = 2x^Tx.$$

Hence $x = 0$. Therefore, $(I+A)$ is nonsingular.

ⓒ A square matrix M is orthogonal if and only if $M^T M = I$. Note that

$$[(I-A)(I+A)^{-1}]^T (I-A)(I+A)^{-1} = (I+A)^{-T}(I-A)^T(I-A)(I+A)^{-1}$$

$$= (I+A^T)^{-1}(I-A^T)(I-A)(I+A)^{-1}$$

$$= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \quad [A=-A^T]$$

$$= (I-A)^{-1}(I+A)(I+A)^{-1}(I-A) \quad [\text{from } ⓐ]$$

$$= I$$

Therefore, $(I-A)(I+A)^{-1}$ is orthogonal.

ⓓ Note that

$$B^T = [(I-A)(I+A)^{-1}]^T = (I+A)^{-T}(I-A)^T$$

$$= (I+A^T)^{-1}(I-A^T) = (I+A^{-1})^{-1}(I-A^{-1}) \quad [\text{since } A^TA = I]$$

$$= (I+A^{-1})^{-1}A^{-1}A(I-A^{-1})$$

$$= [A(I+A^{-1})]^{-1}(A-I)$$

$$= -(I+A)^{-1}(I-A)$$

$$= -(I-A)(I+A)^{-1} \quad [\text{from } ⓐ]$$

$$= -B.$$

③ $A = \begin{bmatrix} a & b & 0 \\ c & d & c \\ 0 & b & a \end{bmatrix}$

ⓐ The matrix A is unitarily diagonalizable if and only if $AA^T = A^TA$.

Note that

$$AA^T = \begin{bmatrix} a^2+b^2 & ac+bd & b^2 \\ ac+bd & c^2+d^2 & ac+bd \\ b^2 & ac+bd & a^2+b^2 \end{bmatrix}$$

and

$$A^TA = \begin{bmatrix} a^2+c^2 & ab+cd & c^2 \\ ab+cd & b^2+d^2 & ab+cd \\ c^2 & ab+cd & a^2+c^2 \end{bmatrix}.$$

Therefore, A is unitarily diagonalizable if and only if

$$a^2+b^2 = a^2+c^2 \quad c^2+d^2 = b^2+d^2 \quad b^2 = c^2 \quad ac+bd = ab+cd.$$

The first three equations result in $b^2 = c^2$, or equivalently $b = \pm c$.

<u>case 1</u>: $(b=c)$ In this case the fourth is already satisfied.

<u>case 2</u>: $(b=-c)$ In this case the fourth boils down to

$$2(a-d)c = 0 \iff a=d \text{ OR } c=0.$$

Therefore, A is unitarily diagonalizable if and only if $(b=c)$ OR $(b=-c\neq 0$ and $a=d)$.

(b) Since $x^T A x = \frac{1}{2} x^T (A + A^T) x$, A is positive definite if and only if so is $A + A^T$.

Note that

$$A + A^T = \begin{bmatrix} 2a & b+c & 0 \\ b+c & 2d & b+c \\ 0 & b+c & 2a \end{bmatrix}.$$

We know that a symmetric matrix is positive definite if and only if all its principal minors are positive. Then, A is positive definite if and only if

$$2a > 0 \qquad \underbrace{\begin{vmatrix} 2a & b+c \\ b+c & 2d \end{vmatrix}}_{4ad - (b+c)^2} > 0 \qquad \underbrace{\begin{vmatrix} 2a & b+c & 0 \\ b+c & 2d & b+c \\ 0 & b+c & 2a \end{vmatrix}}_{8a^2d - 4a(b+c)^2} > 0$$

These result in

$$a > 0 \qquad 4ad - (b+c)^2 > 0 \quad \text{and} \quad 2ad - (b+c)^2 > 0.$$

It follows from the second that $d > 0$. Hence, the second is readily satisfied provided that the other two are satisfied. Therefore, A is positive definite if and only if $a > 0$ and $2ad > (b+c)^2$.

4

(a) Let $x \in \text{span}\{v, Av, \dots, A^{n-1}v\}$. Then,

$$x = \alpha_0 v + \alpha_1 Av + \cdots + \alpha_{n-1} A^{n-1} v$$

for some real number α_i with $i = 0, 1, \dots, n-1$. Note that

$$Ax = \alpha_0 Av + \alpha_1 A^2 v + \cdots + \alpha_{n-2} A^{n-1} v + \alpha_{n-1} A^n v.$$

It follows from Cayley-Hamilton theorem that $A^n v$ belongs to the subspace $\{v, Av, \dots, A^{n-1}v\}$. Since all the other term already belong to the same subspace, we get

$$Ax \in \text{span}\{v, Av, \dots, A^{n-1}v\}.$$

Therefore, this subspace is invariant under A.

(b) Note that

$$\det(M - \lambda I) = -(\lambda^3 + 3\lambda^2 + 3\lambda + 1 - a). = -((\lambda+1)^3 - a)$$

Therefore, Cayley-Hamilton theorem implies that

$$(M+I)^3 = aI.$$

Hence, $(M+I)^{3000} = a^{1000} \cdot I$.

(5) Let A be the given matrix. Note that

$$A^T A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}.$$

and

$$\det(A^T A - \lambda I) = (4-\lambda)\left[(5-\lambda)^2 - 16\right].$$

Hence, the eigenvalues of $A^T A$ are

$$\lambda_1 = 9 \qquad \lambda_2 = 4 \qquad \lambda_3 = 1.$$

Then, the singular values are given by

$$\sigma_1 = 3 \qquad \sigma_2 = 2 \qquad \sigma_3 = 1.$$

First, we compute the eigenvectors of $A^T A$:

For $\lambda_1 = 9$:

$$A^T A - 9I = \begin{bmatrix} -4 & 0 & 4 \\ 0 & -5 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

Then, $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ is the normalized eigenvector corresponding to the eigenvalue $\lambda_1 = 9$

For $\lambda_2=4$:

$$A^TA-4I=\begin{bmatrix}1&0&4\\0&0&0\\4&0&1\end{bmatrix}$$

Then, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is the normalized eigenvector corresponding to the eigenvalue $\lambda_2=4$.

For $\lambda_3=1$:

$$A^TA-I=\begin{bmatrix}4&0&4\\0&3&0\\4&0&4\end{bmatrix}$$

Then, $\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$ is the normalized eigenvector corresponding to the eigenvalue $\lambda_3=1$.

Therefore,

$$V=\begin{bmatrix}1/\sqrt{2}&0&1/\sqrt{2}\\0&1&0\\1/\sqrt{2}&0&-1/\sqrt{2}\end{bmatrix}$$

This yields that

$$u_1=\frac{1}{\sigma_1}Av_1=\begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}\quad u_2=\frac{1}{\sigma_2}Av_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\quad u_3=\frac{1}{\sigma_3}Av_3=\begin{bmatrix}1/\sqrt{2}\\0\\0\\-1/\sqrt{2}\end{bmatrix}.$$

Finally, we need to compute $N(\bar{A}^T)$ to find u_4. Note that

$$N(\bar{A}^T) = N\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Hence, $u_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then, we have

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

as an SVD of A. Then, the best rank-1 approximation can be found as

$$\bar{A} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 0 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3/2 & 0 & 3/2 \end{bmatrix}$$

⑥ Let A be the given matrix. Note that

$$\det(A-\lambda I) = -(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = -(\lambda-1)^3.$$

Also note that

$$A - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \qquad (A-I)^2 = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$(A-I)^3 = 0.$$

Then, we can conclude that the Jordan form should be given by

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that

$$(A-I)^3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \qquad (A-I)^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0.$$

Define

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = (A-I)x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad x_1 = (A-I)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note that Jordan canonical form can be obtained as follows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$